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Working Paper Number 20, 2018
Center for Quantitative Risk Analysis (CEQURA)
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University of Munich

http://www.cequra.uni-muenchen.de

A note on low-dimensional Kalman smoother for systems with lagged states in the measurement equation

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January 10, 2018

Abstract

This note shows that the modified Kalman smoother for state space systems with lagged states in the measurement equation, introduced in Nimark (2015, Economics Letters 127), is in general not minimizing the mean squared error (MSE). We derive the MSE-minimizing Kalman smoother, discuss alternative computationally more efficient smoothing algorithms and compare the competing smoother with regards to the MSE.

JEL Classification: C18, C22, C32.
Keywords: Kalman filter, Kalman smoother, Lagged states.

In this note, we consider state space systems with a lagged state in the measurement equation for which Nimark (2015) derives a modified low-dimensional Kalman filter. Nimark (2015) also states, without a formal derivation, that the filtered state estimates from the modified filter can be plugged into the standard, i.e., unmodified, Kalman smoother of Hamilton (1994). In this paper we show that to use the filtered state estimates from the modified filter, we also need to modify the Kalman smoother to obtain the MSE-minimizing smooth state estimates. That is, the claim that the filtered estimates from Nimark's (2015) modified filter can be plugged into the standard smoother is in general not correct. In what follows, we derive three modified Kalman smoother that all can be used in combination with the modified Kalman filter of Nimark (2015). The first is based on the same principles as the one in Hamilton (1994). The second and third, and computationally more efficient, smoother are a modified version of the smoother of de Jong (1988, 1989) and Kohn and Ansley (1989), and a modified version of the disturbance-smoother-based state smoother of Koopman (1993). Finally, the minimum variance estimator for the smooth states will be compared to the Nimark (2015) smoother.

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1 The state space model

In this note we stick as close as possible to the notation of Nimark (2015) and consider the state space model

$$X_t = AX_{t-1} + Cu_t, \qquad Z_t = D_1 X_t + D_2 X_{t-1} + Ru_t, \tag{1.1}$$

where u_t is a m-dimensional vector of disturbances being multivariate normally distributed with zero mean and the identity as variance-covariance matrix. The observable at time t, Z_t, is a $p \times 1$ vector and the state vector X_t is of dimension $n \times 1$. Similar to Nimark (2015), we use for the conditional expectation and variance the notations

$$X_{t|t-s} = \mathbb{E}(X_t | Z_{1:t-s}, X_{0|0}), \qquad P_{t|t} = \mathbb{E}((X_t - X_{t|t})(X_t - X_{t|t})'),$$

with $Z_{1:t} = (Z_1', \ldots, Z_t')'$ and we initialize the system by $X_0 \sim N(X_{0|0}, P_{0|0})$.

2 The modified Kalman filter

The standard solution to apply the Kalman filter to the state space system (1.1) is obtained by augmenting the state vector with lagged states. A modified Kalman filter, which operates with an n-dimensional state vector, was derived by Nimark (2015). Nimark's (2015) modified Kalman filter can be summarized by the following recursion

$$\tilde{Z}_t = Z_t - \tilde{D} X_{t-1|t-1}, \qquad P_{t|t-1} = AP_{t-1|t-1}A + CC', \tag{2.1}$$

$$X_{t|t} = AX_{t-1|t-1} + K_t \tilde{Z}_t, \qquad P_{t|t} = P_{t|t-1} - K_t F_t K_t', \tag{2.2}$$

with $\tilde{D} = (D_1 A + D_2)$ and where the Kalman gain is given by $K_t = U_t F_t^{-1}$ with

$$U_t = \mathbb{E}(X_t \tilde{Z}_t') = AP_{t-1|t-1}\tilde{D}' + CC'D_1' + CR', \tag{2.3}$$

$$F_t = \mathbb{E}(\tilde{Z}_t \tilde{Z}_t') = \tilde{D} P_{t-1|t-1} \tilde{D}' + (D_1 C + R)(D_1 C + R)'. \tag{2.4}$$

3 On the Kalman smoother for systems with a lagged state in the measurement equation

To derive the updating equations which are purely based on filtered states and not on the observables, Hamilton (1994) uses the following approach.[1] By the formula for updating linear projections (Eq. [4.5.30] in Hamilton (1994)) one gets

$$\mathbb{E}(X_t | X_{t+1}, Z_{1:t}, X_{0|0}) = X_{t|t} + \hat{J}_t (X_{t+1} - X_{t+1|t}),$$

with $\hat{J}_t = P_{t|t} A' P_{t+1|t}^{-1}$. In a next step, Hamilton (1994) argues that $\mathbb{E}(X_t | X_{t+1}, Z_{1:t}, X_{0|0})$ is equal to $\mathbb{E}(X_t | X_{t+1}, Z_{1:T}, X_{0|0})$, as the error

$$X_t - \mathbb{E}(X_t | X_{t+1}, Z_{1:t}, X_{0|0})$$

[1] This state smoothing algorithm goes back to Anderson and Moore (1979) and Rauch et al. (1965).

is uncorrelated with Z_{t+j}, for $0 < j \leq T - t$. While this is true for a standard Kalman filter, as shown in Hamilton (1994), this is (in general) not the case for state space systems with a lagged state in the measurement equation, i.e., in general for state space systems of the form (1.1)

$$\mathrm{Corr}(X_t - \mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0}), Z_{t+1}) \neq 0$$

and therefore

$$\mathbb{E}(X_t|X_{t+1}, Z_{1:t}, X_{0|0}) \neq \mathbb{E}(X_t|X_{t+1}, Z_{1:T}, X_{0|0}). \tag{3.1}$$

As a consequence, the smoother stated in Eq. (4.2) in Nimark (2015)[2]

$$\hat{X}_{t|T} = X_{t|t} + \hat{J}_t(X_{t+1|T} - X_{t+1|t}), \qquad \hat{J}_t = P_{t|t}A'P^{-1}_{t+1|t}, \tag{3.2}$$

is in general not equal to $\mathbb{E}(X_t|Z_{1:T}, X_{0|0})$ as claimed by Nimark (2015). Note that in general the smooth estimate, $\hat{X}_{t|T}$, (Eq. (3.2)) is also not minimizing the MSE to X_t conditional on the complete history of the observables $Z_{1:T}$.

This can be easily verified, e.g., by considering the special case $A = 0_{n\times n}$. Then, by (3.2), we get

$$\hat{X}_{T-1|T} = X_{T-1|T-1} \quad \Rightarrow \quad \mathrm{Var}(X_{T-1} - \hat{X}_{T-1|T}) = P_{T-1|T-1} \tag{3.3}$$

and in contrast for

$$X_{T-1|T} = X_{T-1|T-1} + P_{T-1|T-1}D_2'F_T^{-1}\tilde{Z}_T \tag{3.4}$$

we obtain

$$\mathrm{Var}(X_{T-1} - X_{T-1|T}) = P_{T-1|T-1} - P_{T-1|T-1}D_2'F_T^{-1}D_2P_{T-1|T-1}. \tag{3.5}$$

Both smoother, (3.2) and (3.4), are obviously unbiased and as $P_{T-1|T-1}D_2'F_T^{-1}D_2P_{T-1|T-1}$ is positive semidefinite if F_T is positive semidefinite it follows with (3.3) and (3.5)

$$\begin{aligned}\mathrm{MSE}(X_{T-1|T}) &= \mathrm{tr}(P_{T-1|T-1}) - \mathrm{tr}(P_{T-1|T-1}D_2'F_T^{-1}D_2P_{T-1|T-1}) \\ &\leq \mathrm{tr}(P_{T-1|T-1}) = \mathrm{MSE}(\hat{X}_{T-1|T}),\end{aligned}$$

i.e., the smoother, $\hat{X}_{T-1|T}$, is not the MSE-minimizing estimator of X_{T-1} given the complete history of the observables $Z_{1:T}$.

4 Kalman smoothing algorithms for the modified system

Similar to Hamilton (1994), the MSE-minimizing smoother for the modified system can be obtained using the updating equation for linear projections but with an adaption for systems with a lagged state in the measurement equation. Start by considering the conditional expectation $\mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0})$ and by applying the formula for updating a linear projection (Eq. [4.5.30] in Hamilton (1994))

$$\mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0}) = X_{t|t+1} + \mathbb{E}((X_t - X_{t|t+1})(X_{t+1} - X_{t+1|t+1})')$$

[2] Note that there is a typo in Eq. (4.2) in Nimark (2015), where the index of $\hat{J}$ was $t-1$ instead of t, as in Hamilton (1994).

$$\begin{aligned} &\cdot \mathbb{E}((X_{t+1} - X_{t+1|t+1})(X_{t+1} - X_{t+1|t+1})')^{-1}(X_{t+1} - X_{t+1|t+1}) \\ &= X_{t|t+1} + P'_{t+1,t|t+1} P^{-1}_{t+1|t+1}(X_{t+1} - X_{t+1|t+1}), \end{aligned}$$

where $P_{t+1,t|t+1} = \mathbb{E}((X_{t+1} - X_{t+1|t+1})(X_t - X_{t|t+1})') = AP_{t|t} - K_{t+1}\tilde{D}P_{t|t}$. From the standard theory on state smoothing (see, e.g., Durbin and Koopman (2012)), we get the one-step ahead smooth state as

$$X_{t|t+1} = X_{t|t} + P_{t|t}\tilde{D}'F^{-1}_{t+1}\tilde{Z}_{t+1}.$$

Future observables, Z_{t+j}, for $1 < j \leq T - t$, can be written as

$$Z_{t+j} = \tilde{D}X_{t+j-1} + (D_1C + R)u_{t+j} = \tilde{D}\Big(A^{j-2}X_{t+1} + \sum_{i=2}^{j-1} A^{j-1-i}Cu_{t+i}\Big) + (D_1C + R)u_{t+j},$$

where we use the notational convention that A^0 is the identity and A^n denotes the n-th power of the square matrix A. Therefore, using the same reasoning as in Hamilton (1994), we see that the prediction error

$$X_t - \mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0}) = X_t - X_{t|t+1} - P'_{t+1,t|t+1}P^{-1}_{t+1|t+1}(X_{t+1} - X_{t+1|t+1}) \tag{4.1}$$

is uncorrelated with Z_{t+j} for $1 < j \leq T - t$. This follows because the prediction error (4.1) is by construction uncorrelated with X_{t+1}, and by assumption uncorrelated with $u_{t+j}, u_{t+j-1}, \dots, u_{t+2}$. As a consequence, we get

$$\mathbb{E}(X_t|X_{t+1}, Z_{1:T}, X_{0|0}) = \mathbb{E}(X_t|X_{t+1}, Z_{1:t+1}, X_{0|0}) \tag{4.2}$$

and by applying the law of iterated projections, as Hamilton (1994), we obtain the smooth estimate, $\mathbb{E}(X_t|Z_{1:T}, X_{0|0})$, by projecting (4.2) on $Z_{1:T}$. The smooth estimate is given by

$$X_{t|T} = \mathbb{E}(X_t|Z_{1:T}, X_{0|0}) = X_{t|t+1} + J_t(X_{t+1|T} - X_{t+1|t+1}), \tag{4.3}$$

with $J_t = P'_{t+1,t|t+1}P^{-1}_{t+1|t+1}$.

4.1 MSE of the smooth state

Analogously to Hamilton (1994), by subtracting X_t from Eq. (4.3) and rearranging, we obtain

$$X_t - X_{t|T} + J_tX_{t+1|T} = X_t - X_{t|t+1} + J_tX_{t+1|t+1}. \tag{4.4}$$

Multiplying (4.4) with its transpose and applying the expectation implies

$$P_{t|T} + J_t\mathbb{E}(X_{t+1|T}X'_{t+1|T})J'_t = P_{t|t+1} + J_t\mathbb{E}(X_{t+1|t+1}X'_{t+1|t+1})J'_t, \tag{4.5}$$

where we used $\mathbb{E}((X_t - X_{t|T})X'_{t+1|T}) = 0$ and $\mathbb{E}((X_t - X_{t|t+1})X'_{t+1|t+1}) = 0$. Rearranging (4.5) results in the backward-recursion

$$P_{t|T} = P_{t|t+1} + J_t(P_{t+1|T} - P_{t+1|t+1})J'_t. \tag{4.6}$$

4.2 Computationally more efficient smoother for the modified system

We first reformulate the state space problem (1.1) in a comparable way as the so-called innovation analogue stated in Durbin and Koopman (2012). Using the updating equations of the modified filter (2.1)–(2.2), we get with $L_t = A - K_t\tilde{D}$ and $M_t = C - K_t(D_1C + R)$

$$\tilde{Z}_t = Z_t - Z_{t|t-1} = \tilde{D}\tilde{X}_{t-1} + (D_1C + R)u_t, \tag{4.7}$$

$$\tilde{X}_t = X_t - X_{t|t} = (A - K_t\tilde{D})\tilde{X}_{t-1} + (C - K_t(D_1C + R))u_t = L_t\tilde{X}_{t-1} + M_tu_t. \tag{4.8}$$

As shown by Durbin and Koopman (2012), Z_T is fixed if Z_t and $\tilde{Z}_{t+1}, \ldots, \tilde{Z}_T$ are fixed. Note that the errors $\tilde{Z}_{t+1}, \ldots, \tilde{Z}_T$ are uncorrelated and $\mathbb{E}(\tilde{Z}_j|Z_{1:t}, X_{0|0}) = 0$ for $j = t+1, \ldots, T$. By the formula for updating a linear projection (Eq. [4.5.30] in Hamilton (1994)), it follows for the smooth state

$$X_{t|T} = X_{t|t} + \sum_{j=t+1}^{T} \mathbb{E}(\tilde{X}_t\tilde{Z}_j')\mathbb{E}(\tilde{Z}_j\tilde{Z}_j')^{-1}\tilde{Z}_j, \tag{4.9}$$

where $E(\tilde{Z}_j\tilde{Z}_j')^{-1} = F_j^{-1}$ is the second term of the Kalman gain (2.4). Using Eq. (4.7)–(4.8), we get for $j = t+1, \ldots, T$

$$\mathbb{E}(\tilde{X}_t\tilde{Z}_j') = \mathbb{E}(\tilde{X}_t\tilde{X}_{j-1}')\tilde{D}' + \mathbb{E}(\tilde{X}_tu_j')(D_1C + R)' = P_{t|t}L_{t+1}' \cdots L_{j-1}'\tilde{D}', \tag{4.10}$$

where we apply the notational convention that $L_{t+1}' \cdots L_{j-1}'$ is the identity I_n for $j = t+1$ and L_{t+1}' for $j = t+2$. Inserting Eq. (4.10) into Eq. (4.9) results in the backward recursion

$$X_{t|T} = X_{t|t} + P_{t|t}r_t, \qquad r_t = \tilde{D}'F_{t+1}^{-1}\tilde{Z}_{t+1} + L_{t+1}'r_{t+1}, \tag{4.11}$$

with initial conditions $r_T = 0_{n\times 1}$ and $L_T = 0_{n\times n}$. By the theory for updating linear projections (Eq. [4.5.31] in Hamilton (1994)), we obtain for the variance of the smooth state vector

$$P_{t|T} = P_{t|t} - \sum_{j=t+1}^{T} \mathbb{E}(\tilde{X}_t\tilde{Z}_j')\mathbb{E}(\tilde{Z}_j\tilde{Z}_j')^{-1}\mathbb{E}(\tilde{Z}_j\tilde{X}_t'), \tag{4.12}$$

and by inserting Eq. (4.10) into (4.12), we obtain the backward recursion

$$P_{t|T} = P_{t|t} - P_{t|t}N_tP_{t|t}, \qquad N_t = \tilde{D}'F_{t+1}^{-1}\tilde{D} + L_{t+1}'N_{t+1}L_{t+1}, \tag{4.13}$$

with initial condition $N_T = 0_{n\times n}$. Note that the backward recursion for the smooth state (4.11) and its variance (4.13) are very similar to the smoother proposed in de Jong (1988, 1989) and Kohn and Ansley (1989) but with a modification to be applicable in the context of Nimark's (2015) modified Kalman filter.

An even more efficient fast state smoothing recursion, similar to Koopman (1993), can be obtained by computing the smooth disturbances via the backward recursion

$$u_{t|T} = (D_1C + R)'F_t^{-1}\tilde{Z}_j + M_t'r_t, \tag{4.14}$$

with the recursively defined r_t from (4.11). Then, like Koopman (1993), we obtain via a forward recursion the smooth states as

$$X_{t|T} = AX_{t-1|T} + Cu_{t|T}, \tag{4.15}$$

with initial condition $X_{0|T} = X_{0|0} + P_{0|0} r_0$.

The gains in computational costs of the modified de Jong (1988, 1989) and Kohn and Ansley (1989) smoother (4.11) and the modified Koopman (1993) smoother (4.14)–(4.15) are comparable to the classical case, i.e., without lagged state in the measurement equation.[3] Further, note that the three different recursions for the smooth states (4.3), (4.11) and (4.14)–(4.15) are equivalent, i.e., return the same smooth states $X_{t|T} = \mathbb{E}(X_t | Z_{1:T}, X_{0|0})$. The same applies to the two different recursions for obtaining the variance of the smoother (4.6) and (4.13).

4.3 The MSE of the Nimark (2015) smoother

Nimark (2015) claims that by Hamilton (1994) the variance of the smooth state, $\hat{X}_{t|T}$, is given by $\hat{P}_{t|T} = P_{t|t} + \hat{J}_t(\hat{P}_{t+1|T} - P_{t+1|t})\hat{J}_t'$. As in general $\hat{X}_{t|T} \neq \mathbb{E}(X_t | Z_{1:T}, X_{0|0})$ (see (3.1)–(3.2)), the formula of Hamilton (1994) cannot be directly applied to obtain the variance of $\hat{X}_{t|T}$. The smoother, $\hat{X}_{t|T}$ (Eq. (3.2)), can be rewritten as

$$\hat{X}_{t|T} = X_{t|t} + \sum_{j=t+1}^{T} \hat{J}_t \cdots \hat{J}_{j-1} K_j \tilde{Z}_j.$$

As the errors $\tilde{Z}_j$ are uncorrelated, the variance of $\hat{X}_{t|T}$ can be obtained as

$$P_{t|t} + \sum_{j=t+1}^{T} \Big[\hat{J}_t \cdots \hat{J}_{j-1} K_j F_j K_j' \hat{J}_{j-1}' \cdots \hat{J}_t' - \hat{J}_t \cdots \hat{J}_{j-1} K_j \mathbb{E}(\tilde{Z}_j \tilde{X}_t') - \mathbb{E}(\tilde{X}_t \tilde{Z}_j') K_j' \hat{J}_{j-1}' \cdots \hat{J}_t' \Big]. \tag{4.16}$$

Inserting Eq. (4.10) into Eq. (4.16) results in the backward recursion

$$\begin{aligned} \mathbb{E}((X_t - \hat{X}_{t|T})(X_t - \hat{X}_{t|T})') &= P_{t|t} + \hat{J}_t \hat{N}_t \hat{J}_t' - \hat{J}_t \hat{M}_t P_{t|t} - P_{t|t} \hat{J}_t' \hat{M}_t', \\ \hat{N}_t &= K_{t+1} F_{t+1} K_{t+1}' + \hat{J}_{t+1} \hat{N}_{t+1} \hat{J}_{t+1}', \\ \hat{M}_t &= K_{t+1} \tilde{D} + \hat{J}_{t+1} \hat{M}_{t+1} L_{t+1}, \end{aligned}$$

with initial conditions $\hat{N}_T = 0_{n\times n}$ and $\hat{M}_T = 0_{n\times n}$.

5 Application: ARMA dynamics with measurement error

Data revisions are a typical phenomenon for economic time series. As a consequences researcher and decision maker have to rely on econometric models which are capable to allow or even explicitly model measurement errors. Jacobs and van Norden (2011) propose to use a state space model with a Kalman filter if the "true" signal can be described by a stochastic process, like an ARMA-process, and the signals can only be observed up to a measurement error.[4] In the following we will study ARMA(1,1)-processes, and as a special case a MA(1)-process, with measurement

[3] A discussion of the comparative computational efficiency of the different smoother algorithms in the unmodified case can be found in Section 4.6.1–4.6.2 of Durbin and Koopman (2012) and in Koopman (1993).

[4] In the context of data revisions, Jacobs and van Norden (2011) provide a detailed discussion of measurement errors and their components, like noise, news and spillover effects.

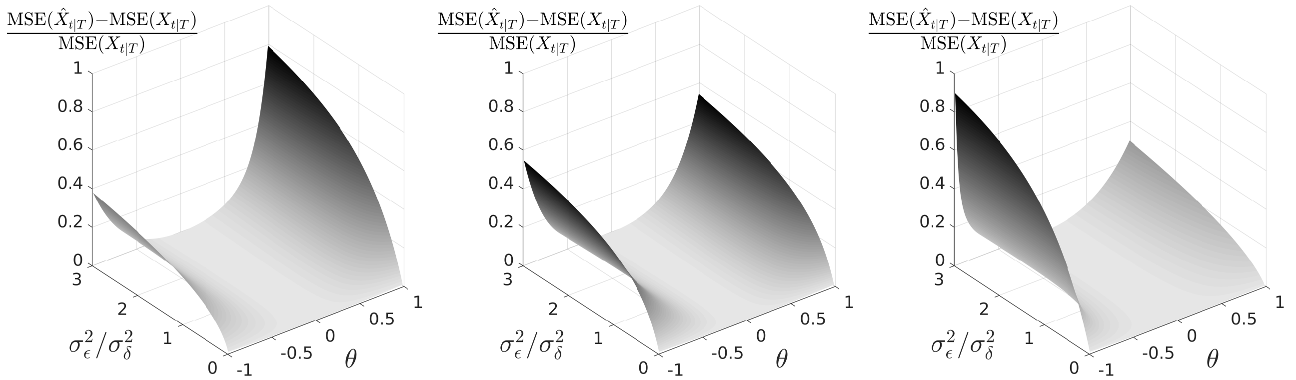

Figure 1: Relative increase in the MSE, $(\mathrm{MSE}(\hat{X}_{T|t}) - \mathrm{MSE}(X_{T|t}))/\mathrm{MSE}(X_{T|t})$, as a function of the MA(1)-parameter, θ_1, and the signal-to-noise ratio, $q = \sigma_\epsilon^2/\sigma_\delta^2$. The left, middle and right plots show the result for the AR(1)-parameters $\phi_1 = -0.5$, 0, 0.9, respectively.

error. Like in Jacobs and van Norden (2011) let $\tilde{y}_t$ be the "true" unobserved value. We assume for $\tilde{y}_t$ an ARMA(1,1)-process

$$\tilde{y}_t = \phi_1 \tilde{y}_{t-1} + \theta_1 \epsilon_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \sigma_\epsilon^2),$$

and an additive measurement error $\delta_t \sim N(0, \sigma_\delta^2)$, so that the observed value is given by $y_t = \tilde{y}_t + \delta_t$. Using a specific state space representation of ARMA-processes given in Hamilton (1994, Eq. [13.1.22]–[13.1.23]) and by applying the theory of Nimark (2015), we get for y_t a state space representation with lagged state in the measurement equation of the form (1.1)

$$X_t = \phi_1 X_{t-1} + \sigma_\epsilon u_{1,t}, \qquad y_t = X_t + \theta_1 X_{t-1} + \sigma_\delta u_{2,t},$$

with a one-dimensional state variable, X_t, and a bivariate disturbance vector $u_t := (u_{1,t}, u_{2,t})'$ being independent standard normally distributed. As the parameters are not time-varying, we can numerically determine the steady state for the Kalman recursion and compute the MSE in the steady state for both smoother.

W.l.o.g. let $\sigma_\epsilon^2 = 1$ be the variance of the disturbance of the ARMA(1,1)-signal, and the variance of the measurement error $\sigma_\delta^2 = \sigma_\epsilon^2/q$ is implicitly given by the signal-to-noise ratio, q. For the AR(1)-parameter we consider the values $\phi_1 = -0.5$, 0, 0.9, for the MA(1)-parameter the range $\theta_1 \in [-0.99, 0.99]$ and for the signal-to-noise ratio the range $q \in [0.01, 3]$. In Figure 1, the left, middle and right plots correspond to the different AR(1)-parameters and the relative difference in the MSE of the two smoother, $(\mathrm{MSE}(\hat{X}_{T|t}) - \mathrm{MSE}(X_{T|t}))/\mathrm{MSE}(X_{T|t})$, is plotted against the MA(1)-parameter and the signal-to-noise ratio. We clearly see that, if $\theta_1 = 0$ both smoother have the same MSE. For all other considered scenarios the smoother of Nimark (2015), $\hat{X}_{T|t}$, is having a larger MSE than the MSE-minimizing smoother, $X_{T|t}$. In the most extreme shown case, $\phi_1 = 0.9$, $\theta_1 = -0.99$, $q = 3$, the MSE of the smoother $\hat{X}_{T|t}$ is 89.46% larger than for the MSE-minimizing smoother.

6 Conclusion

In this note we derive the MSE-minimizing smoother for the modified Kalman filter of Nimark (2015) for state space systems with a lagged state in the measurement equation. We demonstrate that the smoother of Nimark (2015) is not minimizing the MSE. Furthermore, we present computationally more efficient smoothing algorithms for the modified system. The MSE-minimizing smoother for the modified system can also be used in combination with the simulation smoother of Durbin and Koopman (2002) as suggested by Nimark (2015).

Accompanying `MATLAB` code is available at https://github.com/MalteKurz/SSMwLS.

Acknowledgments

The author thanks Stefan Mittnik for helpful discussions and he is particularly thankful to Kristoffer P. Nimark for valuable comments, suggestions and discussions which helped to greatly improve this paper.

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